Announcements

1) Any questions?

2) Frobenius norm?

3) HW #1 due Thursday

Definition: Let $A \in \mathbb{C}^{m \times n}$.

The **Frobenius norm** of A,

denoted by $\|A\|_F$,

is the 2-norm of A,

considered as an element

of $\mathbb{C}^{mn}$.

Graphing with Matlab

plot($X_1, Y_1, X_2, Y_2, \ldots$)

Plots Y_1 against X_1,

Y_2 against X_2, etc.,

on the same axes.

To plot point, plot

x-coordinates as a vector

followed by y-coordinates (vector)

To get points instead of lines, include 'o' after the last y value.

To get powers of a variable x, write "x.*x" for x^2, "x.*x.*x" for x^3, etc.

Recall definition (unitary)

Let $Q \in \mathbb{C}^{m \times m}$. Then

Q is unitary if

$$\boxed{Q^* Q = I_m}.$$

Example 1: (2x2 unitary)

$$x = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}.$$

$$y = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

$\|x\|_2 = \|y\|_2 = 2$. Then

$$Q = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$ is

unitary!

Properties of Unitary Matrices

Let $Q \in M_n(\mathbb{C})$. Then the following properties are equivalent:

1) Q is unitary

2) $Q^* = Q^{-1}$

3) $\boxed{\|Qx\|_2 = \|x\|_2 \quad \forall x \in \mathbb{C}^n,}$

4) $QQ^* = I_n$

Example 2: $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$\|A\|_\infty$ = maximal row sum (with absolute values)

$= 3$

$\|A\|_1$ = maximal column sum (with absolute value)

$= 4$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\|A\|_F = \sqrt{1^2 + 2^2 + 0^2 + 2^2}$$

$$= 3$$

$$\|A\|_2 = \sqrt{\|A^*A\|_2}$$

$$A^*A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

$$\|A^*A\|_2 = \frac{1}{2}\left(9+\sqrt{65}\right)$$

So

$$\|A\|_2 = \sqrt{\frac{1}{2}\left(9+\sqrt{65}\right)}$$

The Singular Value Decomposition

Not every $A \in \mathbb{C}^{n \times n}$ can be diagonalized - but if it can ...

Example 3: $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$

Eigenvalues of A are

$$\lambda_1 = \frac{1}{2}(3+\sqrt{5})$$

$$\lambda_2 = \frac{1}{2}(3-\sqrt{5})$$

$$A = Q D Q^*$$

where $D = \begin{bmatrix} \frac{1}{2}(3-\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(3+\sqrt{5}) \end{bmatrix}$

where D is diagonal

and

$$Q = \frac{2\begin{bmatrix} \frac{i}{2}(-1-\sqrt{5}) & -1 \\ 1 & \frac{i}{2}(1+\sqrt{5}) \end{bmatrix}}{\sqrt{10+2\sqrt{5}}}$$

is unitary

In this case, since A is unitarily diagonalizable, the 2-norm of A is the larger of the absolute values of its eigenvalues,

So $\boxed{\|A\|_2 = \frac{1}{2}(3+\sqrt{5})}$

Theorem: (unitary diagonalization)

If $A \in \mathbb{C}^{m \times m}$ satisfies $A = A^*$, then A is unitarily diagonalizable;

i.e., $\exists$ diagonal matrix $D \in \mathbb{C}^{m \times m}$ and Q unitary, $Q \in \mathbb{C}^{m \times m}$, with

$$\boxed{A = QDQ^*}$$

Note: $(A^*A)^* = A^*(A^*)^*$

$= A^*A$

for any $A \in \mathbb{C}^{m\times n}$, so

A^*A is always unitarily diagonalizable.